

# Neutrino flavor instabilities in a time-dependent supernova model

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## Abstract

A dense neutrino medium such as that inside a core-collapse supernova can experience collective flavor conversion or oscillations because of the neutral-current weak interaction among the neutrinos. This phenomenon has been studied in a restricted, stationary supernova model which possesses the (spatial) spherical symmetry about the center of the supernova and the (directional) axial symmetry around the radial direction. Recently it has been shown that these spatial and directional symmetries can be broken spontaneously by collective neutrino oscillations. In this letter we analyze the neutrino flavor instabilities in a time-dependent supernova model. Our results show that collective neutrino oscillations start at approximately the same radius in both the stationary and time-dependent supernova models unless there exist very rapid variations in local physical conditions on timescales of a few microseconds or shorter. Our results also suggest that collective neutrino oscillations can vary rapidly with time in the regimes where they do occur which need to be studied in time-dependent supernova models.

**Keywords:** neutrino oscillations, core-collapse supernova

## 1. Introduction

Neutrinos are essential to the thermal, chemical and dynamical evolution of the early universe and some of the compact objects such as the proto-neutron star inside a core-collapse supernova (SN). Whenever there is a difference between the energy spectra and/or fluxes of the electron-flavor neutrino/antineutrino and other neutrino species, the flavor conversion or oscillations among different neutrino flavors can also have important impacts on nucleosynthesis and other physics inside these hot and dense astrophysical environments.

In the absence of collision the flavor evolution of the neutrino obeys the Liouville equation [1–3]

$$\partial_t \rho + \hat{\mathbf{v}} \cdot \nabla \rho = -i[H_{\text{vac}} + H_{\text{mat}} + H_{\nu\nu}, \rho], \quad (1)$$

where  $\hat{\mathbf{v}}$  is the velocity of the neutrino,  $\rho(t, \mathbf{x}, \mathbf{p})$  is the (Wigner-transformed) flavor density matrices of the neutrino which depends on time  $t$ , position  $\mathbf{x}$  and neutrino momentum  $\mathbf{p}$ ,  $H_{\text{vac}}$  is the standard vacuum Hamiltonian, and  $H_{\text{mat}}$  and  $H_{\nu\nu}$  are the matter and neutrino potentials, respectively. The neutrino potential in Eq. (1) takes the following form [4–6]:

$$H_{\nu\nu} = \sqrt{2}G_F \int \frac{d^3 p'}{(2\pi)^3} (1 - \hat{\mathbf{v}} \cdot \hat{\mathbf{v}}') [\rho(t, \mathbf{x}, \mathbf{p}') - \bar{\rho}(t, \mathbf{x}, \mathbf{p}')], \quad (2)$$

where  $G_F$  is the Fermi coupling constant, and  $\bar{\rho}$  is the density matrix of the antineutrino. Because the neutrino potential couples neutrinos of different momenta, a dense neutrino medium can oscillate in a collective manner (see, e.g., [7–26]; see also [27] for a review).

Eq. (1) poses a challenging 7-dimensional problem (not taking into account the dimensions in neutrino flavors), and it has never been solved in its full form. In previous studies various

simplifications have been made so that a numerical or analytic solution to this equation can be found. For neutrino oscillations in SNe a commonly used model is the (neutrino) Bulb model [13]. In this model a spatial spherical symmetry around the center of the SN is imposed so that it has only one spatial dimension. An additional directional axial symmetry around the radial direction is imposed to make the model self-consistent which reduces the number of momentum dimensions to two. One also imposes the time translation symmetry because the timescale of neutrino oscillations is much shorter than those in neutrino emission or dynamic evolution in SNe. However, it has been shown in a series of recent studies that both the spatial and directional symmetries can be broken spontaneously by collective neutrino oscillations if they are not imposed [28–36] (see also [37] for a short review). In both cases small deviations from the initial symmetric conditions are amplified by the symmetry-breaking oscillation modes which can occur closer to the neutrino sphere than the symmetry-preserving modes do.

It is natural to wonder if collective neutrino oscillations can also break the time-translation symmetry spontaneously in SNe [28]. If they do, then a time-independent SN model may not accurately describe the neutrino oscillation phenomenon in SNe even though the typical timescale of the variation in the neutrino emission is much longer than that of neutrino oscillations. In this letter we analyze the neutrino flavor stability in a time-dependent SN model which should provide some interesting insights to this question.

## 2. Time-dependent neutrino Bulb model

We will focus on the potential differences between the results obtained from the time-dependent and stationary SN models. Therefore, we will employ the time-dependent Bulb model

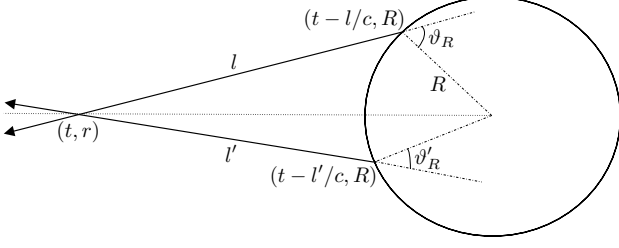


Figure 1: The geometric picture of the time-dependent (neutrino) Bulb model for supernova. Two neutrinos emitted from the neutrino sphere of radius  $R$  with emission angles  $\vartheta_R$  and  $\vartheta'_R$  and at time  $t - l/c$  and  $t - l'/c$  meet each other at radius  $r$  and time  $t$ , where  $l$  and  $l'$  are the distances by which the two neutrinos have traveled from the neutrino sphere to their meeting point, respectively.

which has the same spatial spherical symmetry and the directional axial symmetry as in the conventional Bulb model. Unlike the conventional stationary Bulb model, however, we will not assume that the emission and flavor evolution of the neutrinos are time-independent (see Fig. 1). For simplicity, we will consider the mixing between two active flavors, the  $e$  and  $x$  flavors, with the latter being the linear superposition of the  $\mu$  and  $\tau$  flavors. We also assume a small vacuum mixing angle  $\theta_v \ll 1$ .

It is convenient to use the vacuum oscillation frequency

$$\omega = \pm \frac{|\Delta m^2|}{2E} \quad (3)$$

to label the neutrino and antineutrino with energy  $E$ , where  $\Delta m^2$  is the neutrino mass-squared difference, and the plus and minus signs apply to the neutrino and the antineutrino, respectively. We define reduced neutrino density matrix

$$\varrho(t; r; \omega, u) \propto \begin{cases} \rho & \text{if } \omega > 0, \\ \bar{\rho} & \text{if } \omega < 0 \end{cases} \quad (4)$$

with normalization

$$\text{tr} \varrho = 1, \quad (5)$$

where  $u = \sin^2 \vartheta_R$  with  $\vartheta_R$  being the emission angle of the neutrino on the neutrino sphere (see Fig. 1), and  $r$  is the radial distance from the center of the SN.

The equation of motion (EoM) for the (reduced) density matrix  $\varrho$  can be written as

$$i(\partial_t + v_u \partial_r) \varrho = [H_{\text{vac}} + H_{\text{mat}} + H_{\nu\nu}, \varrho], \quad (6)$$

where

$$v_u(r) = \sqrt{1 - \left(\frac{R}{r}\right)^2} u \quad (7)$$

is the radial velocity of the neutrino. In the weak interaction basis the standard vacuum Hamiltonian and the matter potential are

$$H_{\text{vac}} \approx -\frac{\eta\omega}{2} \sigma_3 = -\frac{\eta\omega}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (8)$$

and

$$H_{\text{mat}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} G_F n_e & 0 \\ 0 & 0 \end{bmatrix}, \quad (9)$$

respectively, where  $\eta = +1$  and  $-1$  for the normal (neutrino mass) hierarchy (NH, i.e. with  $\Delta m^2 > 0$ ) and the inverted hierarchy (IH,  $\Delta m^2 < 0$ ), respectively, and  $n_e$  is the net electron number density.

In this letter we assume that the number flux  $F_{\nu_\alpha/\bar{\nu}_\alpha}(E, \vartheta_R)$  of the neutrino/antineutrino in flavor  $\alpha$  ( $\alpha = e, x$ ) is time independent [38]. We define the distribution function of the neutrino emission to be

$$g(\omega, u) \propto \left| \frac{dE}{d\omega} \right| \times \begin{cases} (F_{\nu_e} + F_{\nu_x}) & \text{if } \omega > 0, \\ -(F_{\bar{\nu}_e} + F_{\bar{\nu}_x}) & \text{if } \omega < 0 \end{cases} \quad (10)$$

with normalization conditions

$$\int_0^\infty d\omega \int_0^1 \frac{du}{2} g(\omega, u) = 1, \quad (11a)$$

$$\int_{-\infty}^0 d\omega \int_0^1 \frac{du}{2} g(\omega, u) = -\frac{N_\nu^{\text{tot}}}{N_{\bar{\nu}}^{\text{tot}}}, \quad (11b)$$

where

$$N_\nu^{\text{tot}} = \int_0^\infty dE \int_0^1 \frac{du}{2} (F_{\nu_e} + F_{\nu_x}), \quad (12a)$$

$$N_{\bar{\nu}}^{\text{tot}} = \int_0^\infty dE \int_0^1 \frac{du}{2} (F_{\bar{\nu}_e} + F_{\bar{\nu}_x}) \quad (12b)$$

are the total number luminosities of the neutrino and antineutrino (i.e. the number of neutrinos or antineutrinos emitted by the whole neutrino sphere per unit time), respectively. The opposite signs of  $g(\omega, u)$  for the neutrino and antineutrino in Eq. (10) take into account their different contributions to the neutrino potential in Eq. (1). In the Bulb model the neutrino potential can be written as

$$H_{\nu\nu}(t; r; u) = \frac{\sqrt{2} G_F N_\nu^{\text{tot}}}{4\pi r^2} \int_{-\infty}^\infty d\omega' \int_0^1 \frac{du'}{v_{u'}} (1 - v_u v_{u'}) \times g(\omega', u') \varrho(t; r; \omega', u'). \quad (13)$$

Because collective neutrino oscillations usually occur in the regime  $R/r \ll 1$  in the Bulb model, we will take the large-radius approximation [39]

$$v_u(r) \approx 1 - \left(\frac{R}{r}\right)^2 \frac{u}{2}. \quad (14)$$

In this approximation,

$$H_{\nu\nu}(t; r; u) \approx \mu \int \left( \frac{u + u'}{2} \right) g' \varrho' d\Gamma', \quad (15)$$

where all the primed quantities are functions of  $u'$  and  $\omega'$ , e.g.,  $\varrho' = \varrho(t; \omega', u'; r)$ ,

$$\mu(r) = \frac{\sqrt{2} G_F N_\nu^{\text{tot}}}{4\pi R^2} \left( \frac{R}{r} \right)^4 \quad (16)$$

is the strength of the neutrino potential at radius  $r$ , and

$$\int d\Gamma' \equiv \int_{-\infty}^\infty d\omega' \int_0^1 du'. \quad (17)$$

### 3. Linear regime

In the regime where no significant flavor transformation has occurred, the linear flavor-stability analysis is applicable [40]. In this regime the neutrino density matrices take the form

$$\varrho(t; r; \omega, u) \approx \frac{\varrho_{ee} + \varrho_{xx}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\varrho_{ee} - \varrho_{xx}}{2} \begin{bmatrix} 1 & \epsilon \\ \epsilon^* & -1 \end{bmatrix}, \quad (18)$$

where  $\varrho_{ee}(\omega, u)$  and  $\varrho_{xx}(\omega, u)$  are the probabilities for the neutrino (or antineutrino) to be in the  $e$  and  $x$  flavors, respectively, and  $|\epsilon(t; r; \omega, u)| \ll 1$ . Here in the spirit of flavor-stability analysis we have assumed that  $\varrho$  are approximately constant. At the onset of collective neutrino oscillations  $\epsilon$  grow exponentially. If  $\epsilon$  has strong time dependence, the time translation symmetry is broken spontaneously by collective neutrino oscillations.

Keeping only the terms up to  $O(\epsilon)$  in Eq. (6) we obtain

$$i(\partial_t + v_u \partial_r) \epsilon \approx (-\eta\omega + \lambda + C) \epsilon - \frac{\mu}{2} \int (u + u') (\varrho'_{ee} - \varrho'_{xx}) g' \epsilon' d\Gamma', \quad (19)$$

where

$$C(u, \mu) = \frac{\mu}{2} \int (u + u') (\varrho'_{ee} - \varrho'_{xx}) g' d\Gamma'. \quad (20)$$

Defining

$$\epsilon_{\varpi}(r; \omega, u) = \int_{-\infty}^{\infty} \epsilon(t; r; \omega, u) e^{i\varpi t} dt \quad (21)$$

we can rewrite Eq. (19) in frequency space as

$$i\partial_r \epsilon_{\varpi} \approx \left\{ -\eta\omega + (\lambda - \varpi) \left[ 1 + \left( \frac{R}{r} \right)^2 \frac{u}{2} \right] + C \right\} \epsilon_{\varpi} - \frac{\mu}{2} \int (u + u') (\varrho'_{ee} - \varrho'_{xx}) g' \epsilon'_{\varpi} d\Gamma'. \quad (22)$$

We note that frequency  $\varpi$  in the above equations represents the temporal variation of the neutrino flavor quantum state at given radius  $r$ , and it shall not be confused with the vacuum oscillation frequency  $\omega$  which is determined by the energy of the neutrino.

We also note that the starting point of collective oscillations is determined by the comparison of the dispersion in each term of the neutrino propagation Hamiltonian in Eq. (1) with the overall strength of the neutrino potential. The spreads in the vacuum Hamiltonian  $H_{\text{vac}}$  and the neutrino potential  $H_{\nu\nu}$  are dominated by variations in vacuum oscillation frequency  $\omega$  and trajectory parameter  $u$ , respectively. Therefore, we have taken  $v_u \approx 1$  for these terms in Eq. (22) as in Ref. [39]. For the matter potential  $H_{\text{mat}}$ , however, the lowest order term of  $v_u^{-1} H_{\text{mat}}$  in the large-radius expansion is the same for all neutrinos and does not suppress collective oscillations [41]. In Eq. (22) we have included its next-order term which can suppress collective oscillations if a very large matter potential is present [39].

Eq. (22) is the same as that for neutrino oscillations in the stationary Bulb model except with replacement  $\lambda \rightarrow \lambda - \varpi$ . The

flavor-stability analysis of this model has been carried out in details in Ref. [40] which we shall not repeat here. The essence of this analysis is to find out all the collective oscillation solutions to Eq. (22) which are of the form

$$\epsilon_{\varpi} = Q_{\varpi} e^{-i\Omega_{\varpi} r}, \quad (23)$$

where  $Q_{\varpi}(\omega, u)$  is independent of  $r$ , and  $\Omega_{\varpi}(\lambda, \mu)$  is the collective oscillation frequency. If

$$\kappa_{\varpi} = \text{Im}(\Omega_{\varpi}) \quad (24)$$

is positive, there exists a flavor instability, and  $\epsilon_{\varpi}$  will grow exponentially in terms of  $r$  which can lead to significant flavor transformation. If there exist multiple unstable modes, the unstable mode with the largest exponential growth rate  $\kappa_{\varpi}^{\text{max}}$  will eventually dominate.

### 4. Results and Discussion

We analyzed the flavor instabilities in the time-dependent Bulb model with two sets of neutrino spectra. In the first case we assume the same single-energy spectrum as in Ref. [34] in which all neutrinos and antineutrinos have the same vacuum oscillation frequency  $\omega_0 = 0.68 \text{ km}^{-1}$  and the number fluxes are  $N_{\nu_e} = 1.25 \times 10^{56} \text{ s}^{-1}$ ,  $N_{\bar{\nu}_e} = 8.32 \times 10^{55} \text{ s}^{-1}$  and  $N_{\nu_x/\bar{\nu}_x} = 5.20 \times 10^{55} \text{ s}^{-1}$ . In the second case we assume the same Fermi-Dirac spectra as in Refs. [22, 40] which have degeneracy parameters  $\eta_{\nu_e} = 3.9$ ,  $\eta_{\bar{\nu}_e} = 2.3$  and  $\eta_{\nu_x/\bar{\nu}_x} = 2.1$ , average energies  $\langle E_{\nu_e} \rangle = 9.4 \text{ MeV}$ ,  $\langle E_{\bar{\nu}_e} \rangle = 13.0 \text{ MeV}$ ,  $\langle E_{\nu_x/\bar{\nu}_x} \rangle = 15.8 \text{ MeV}$ , and luminosities  $L_{\nu_e} = 4.1 \times 10^{51} \text{ erg s}^{-1}$ ,  $L_{\bar{\nu}_e} = 4.3 \times 10^{51} \text{ erg s}^{-1}$ ,  $L_{\nu_x/\bar{\nu}_x} = 7.9 \times 10^{51} \text{ erg s}^{-1}$ . In both cases we assume a neutrino sphere of radius  $R = 10 \text{ km}$  and mass-squared difference  $\Delta m^2 = -2.4 \times 10^{-3} \text{ eV}^2$ , i.e. with an inverted neutrino mass hierarchy.

In Fig. 2 we plot  $\kappa_{\varpi}^{\text{max}}$  for a few frequency modes as functions of neutrino potential strength  $\mu$  [see Eq. (16)] assuming that the matter density is not large enough to suppress collective oscillations (i.e.  $v^{-1}\lambda \approx \lambda$  is valid). In both cases both the instability region and  $\kappa_{\varpi}^{\text{max}}$  are about the same for the frequency modes with  $|\varpi| \lesssim 100 \text{ km}^{-1}$ . This is not a coincidence. Compared to the stationary model, the time-dependent model has a new term in Eq. (22)

$$\frac{\varpi}{v_u} \approx \varpi + \left( \frac{R}{r} \right)^2 \frac{u\varpi}{2}. \quad (25)$$

The first term in the above equation changes only the real part of the collective oscillation frequency  $\Omega_{\varpi}$  and has no impact on the flavor stability. The second term depends on the neutrino trajectory and has a spread  $\Delta\varpi \sim (R/r)^2 |\varpi|$ . It becomes important only when

$$\Delta\varpi \gtrsim \omega_0, \quad (26)$$

where  $\omega_0 \sim 1 \text{ km}^{-1}$  is the typical vacuum oscillation frequency (and also the spread of  $\omega$ ) of supernova neutrinos with the atmospheric mass-squared difference. In both cases collective neutrino oscillations occur at  $r \sim 10R$  which implies that the stability condition of the frequency modes with  $|\varpi| \lesssim 100 \text{ km}^{-1}$  are about the same.

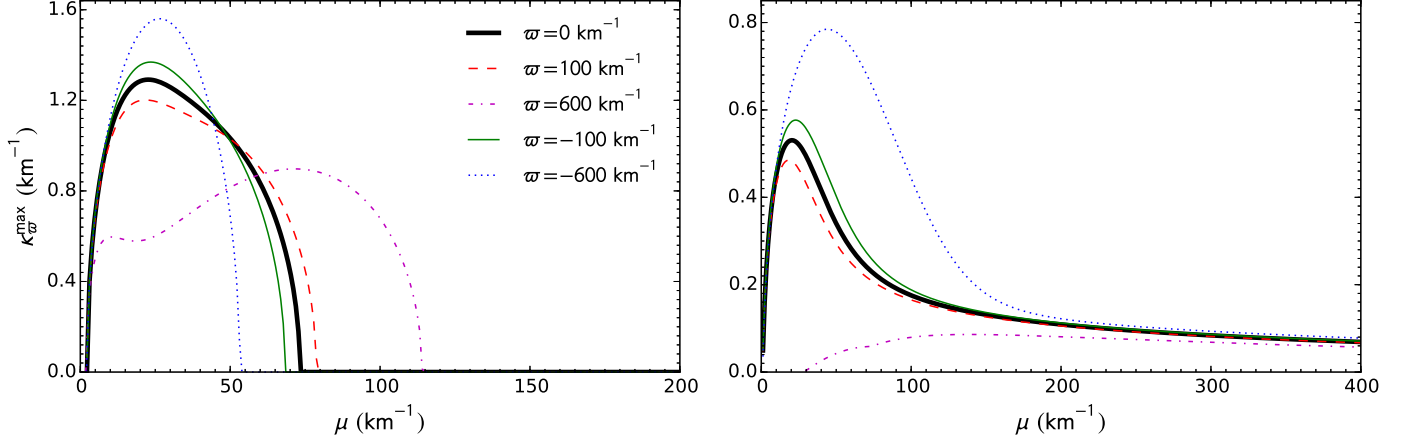


Figure 2: The maximum exponential growth rates  $\kappa_{\omega}^{\max}$  of a few Fourier modes with various frequencies  $\omega$  (as labeled) as functions of the strength of the neutrino potential  $\mu$  in the neutrino Bulb model. The left panel uses a single-energy spectrum [34], and the right panel a continuous spectrum [22, 40]. The peaks of  $\kappa_{\omega=0}^{\max}(\mu)$  correspond to distances 107 km (left) and 149 km (right) from the center of the SN. The matter density is assumed to be not large enough to suppress collective neutrino oscillations.

The above arguments can be generalized to the scenarios where collective oscillations occur close to the neutrino sphere (but not too close so that  $r - R \ll R$ ) because of, e.g., spatial inhomogeneous oscillation modes [32, 33, 35, 36] or different angular distributions for the neutrino fluxes in different flavors [29, 42]. In these scenarios the spread in  $v_u^{-1}\omega$  is of the same order as  $\omega$  itself, and the frequency modes with  $|\omega| \lesssim 1 \text{ km}^{-1} \approx (3 \mu\text{s})^{-1}$  should have the same stability condition. These arguments also apply in the presence of a large matter density because the comparison between  $\Delta\omega$  and  $\omega_0$  is not affected by the presence of the matter potential.

We note that there exists a causality constraint in the time-dependent Bulb model. Suppose that there is a temporary change in the neutrino fluxes on one side of the neutrino sphere which lasts for a time interval  $\Delta t$ . Because it takes at least  $\Delta t' \sim R$  for this change to propagate throughout the proto-neutron star, the assumption of the spherical symmetry implies that the inequality  $\Delta t \gtrsim R$  must hold. Therefore, only the oscillation modes of frequencies

$$\omega \lesssim R^{-1} \sim (10 \text{ km})^{-1} \approx (30 \mu\text{s})^{-1} \quad (27)$$

are allowed in the spherical Bulb model. From the above discussion we conclude that there should be no significant difference between the flavor stability conditions in the time-dependent and stationary Bulb models. For more general time-dependent SN models, collective neutrino oscillations should occur at approximately the same radius as in the corresponding stationary models unless there exist very rapid variations in local physical conditions on the timescales of a few microseconds or shorter.

Meanwhile, the fact that the frequency modes with  $|\omega| \lesssim 1 - 100 \omega_0$  all have similar instability regions also implies that the time-translation symmetry can indeed be broken spontaneously by collective neutrino oscillations in the Bulb model, and that neutrino oscillations can have a strong time dependence once collective oscillations begin. As a result, there may exist qualitative differences between neutrino oscillations in time-dependent and stationary supernova models.

## Acknowledgments

We thank S. Shashank for useful discussions. We appreciate the hospitality of INT/UW where part of this work was done. This work was supported by DOE EPSCoR grant #de-sc0008142 at UNM.

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